Recent progress in finite density lattice QCD towards high density

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Exploration for QCD phase diagram

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Longstanding issues towards the understanding of origin of baryonic matters

- where is QCD critical endpoint ?
- Is chiral symmetry restored in nuclei ?
- What is the EoS inside neutron stars ?

This is challenging issue both for theory and experiment

- accessible range on the diagram, and measurable quantity is limited in experiments.

- sign problem occurs in finite density LQCD

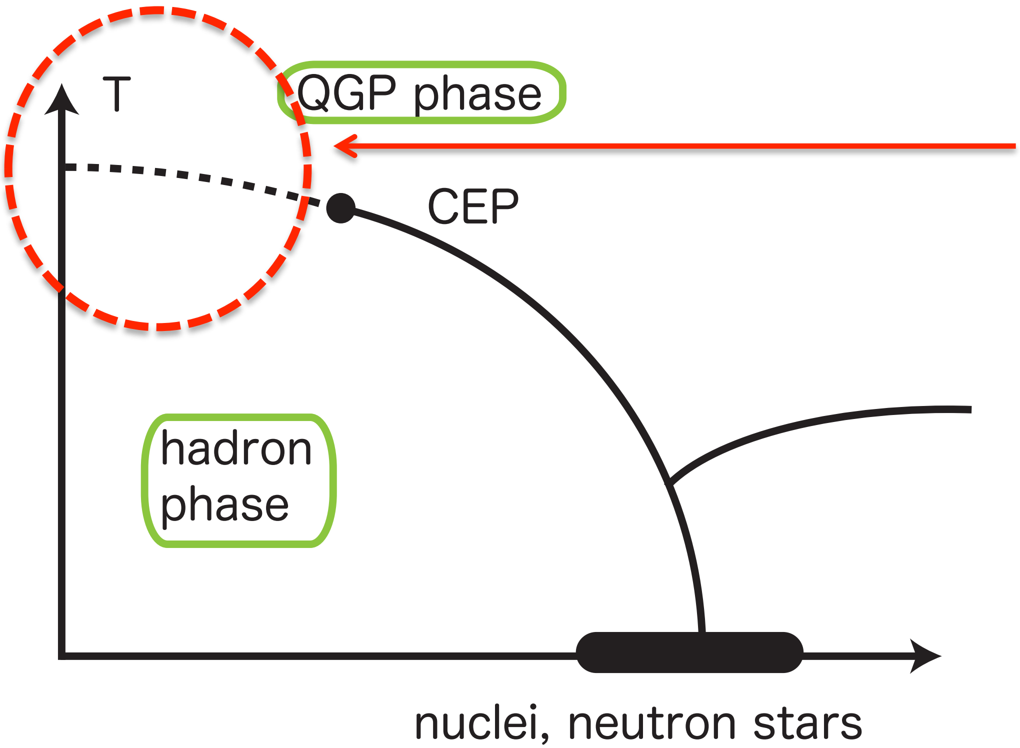
- although many model studies
  - large uncertainty for the location of CEP[e.g. Stephaov:hep-lat/0701002]
  - puzzle in NS with twice solar mass[Demorest, 1010.5788]

A reliable result is necessary to overcome the present situation

How is the situation of lattice QCD ?

MC-based approaches to finite density

– reweighting, Taylor expansions, imaginary chemical potential, etc,

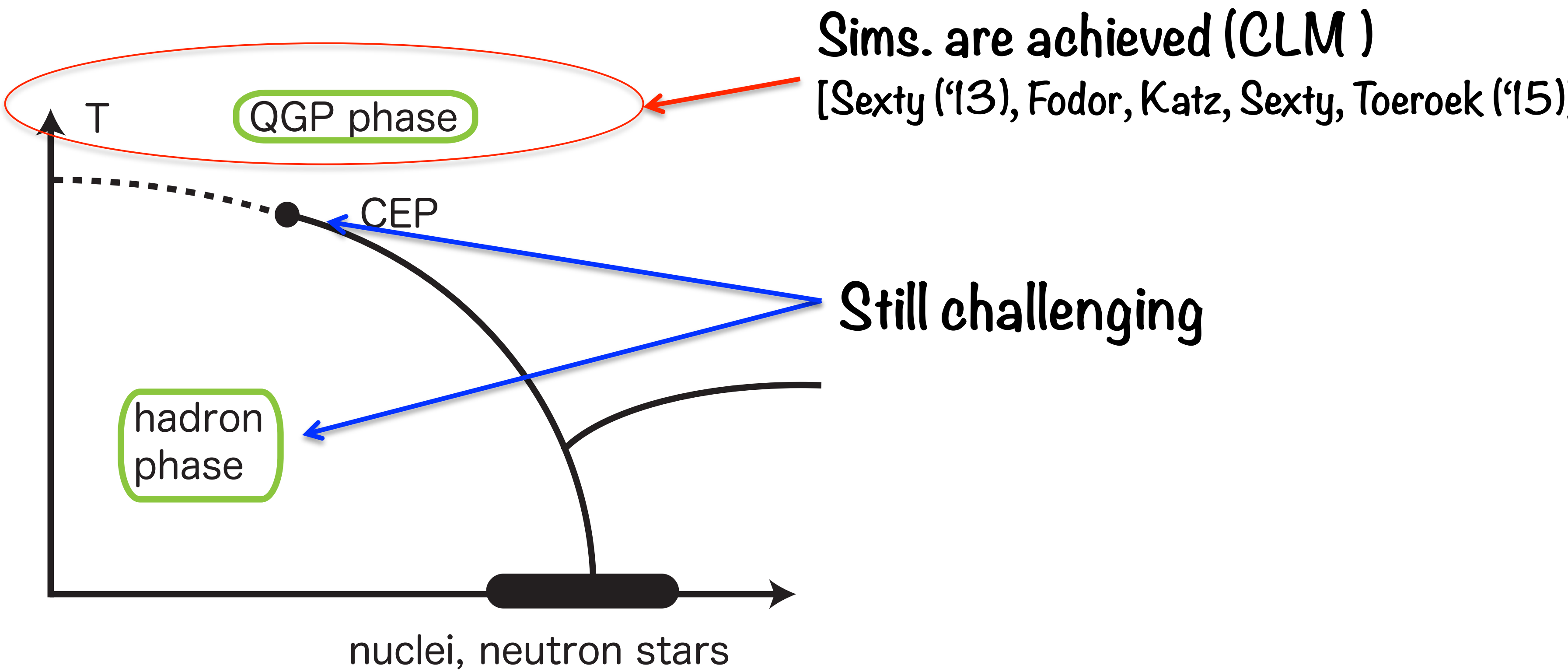


pseudo critical line
EoS, cumulants , ..

New approaches

to go beyond the limit of conventional approaches

e.g. Complex Langevin method(CLM), Lefshetz thimble, tensor networks ...



Purpose of this talk: introduction of recent progress of finite density lattice QCD

1. MC-based approaches: where is the limitation ?

2. complex Langevin to QCD in QGP phase

3. CLM to hadron phase:
   – what is difficulty ?

MC-BASED APPROACH
WHERE IS THE LIMITATION ?

Finite density lattice QCD and sign problem

$$Z(\mu) = \int \mathcal{D}U (\det \Delta(\mu))^{N_f} e^{-S_G}$$

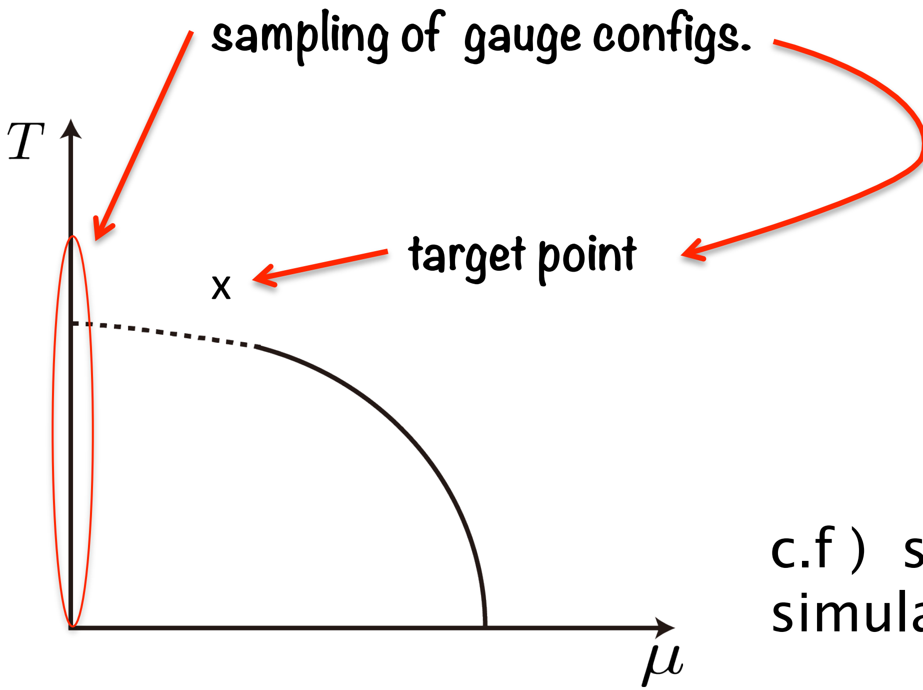
O(V)-dimensional integral

$$\Delta(\mu) = \gamma_\nu D_\nu + m + \gamma_4 \mu$$

- in usual LQCD simulations(μ=0)
  - importance sampling is to sample configs dominating path integral.

- *At nonzero μ, the importance sampling is not available*
  - det Δ(μ) is complex at nonzero μ

Some ideas to access to nonzero μ using gauge configs. obtained at μ=0



- Taylor expansion
- reweighting
- canonical approaches
- imaginary μ

c.f) study a gas at 500 K from simulations at 300K

(Imaginary/isospin chemical potential are also used to generate gauge configs.)

Taylor expansions w.r.t μ

- e.g. free energy

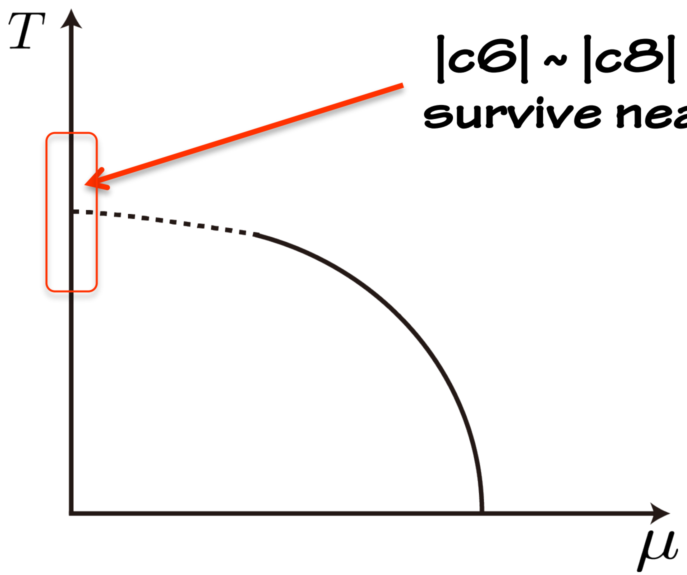
$$-\frac{f(\mu)}{T^4} = c_0 + c_2\left(\frac{\mu}{T}\right)^2 + \cdots$$

- c_n are defined at μ =0, and calculable in ordinary LQCD simulations
- available for any differentiable observables

- in the context of HIC experiments
  - comparison of c_n with cumulants in BES experiments

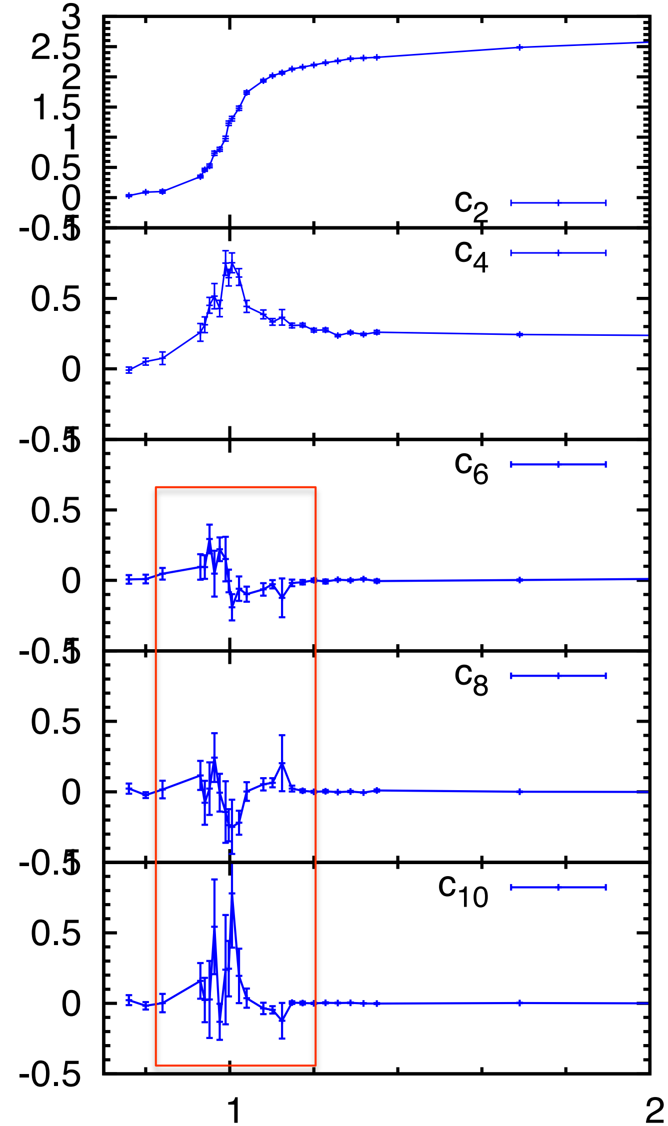
Taylor expansion is reliable for $\mu/T < 1$ for $T \sim Tc$.

Larger μ, larger (uncontrollable) systematic errors



|c6| ~ |c8| ~ |c10|
survive near Tc

$c_n(T)$ vs T/Tc



KN, Nakamura, JHEP1204, 092(2012).

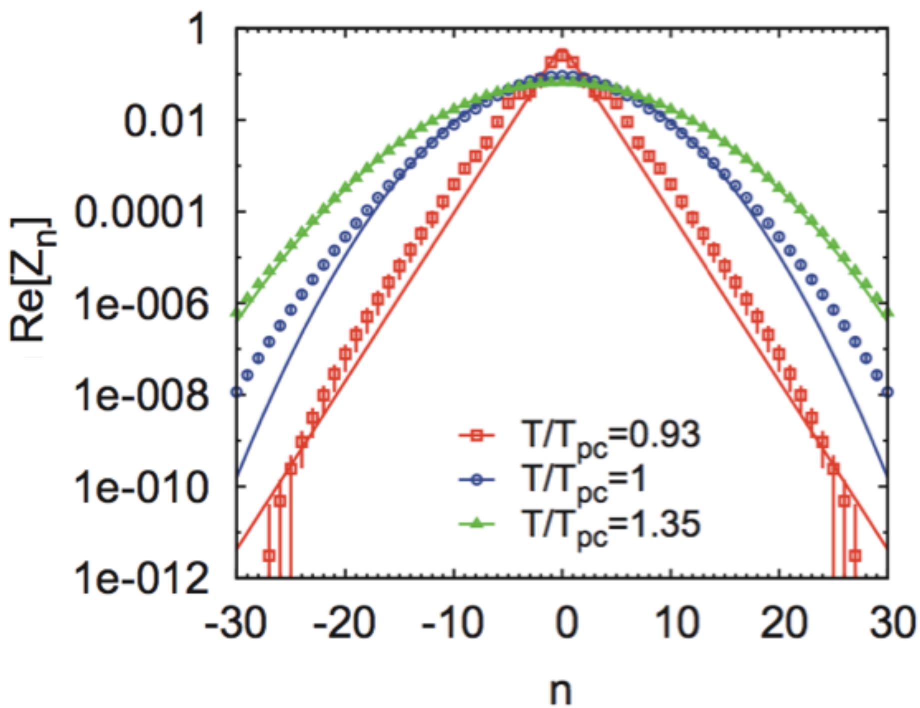
Canonical approach

$$Z(\mu) = \mathrm{tr} e^{-(\hat{H}-\mu\hat{N})/T}$$

$$= \sum_{n=-N}^{N} Z_n e^{n\mu/T}$$

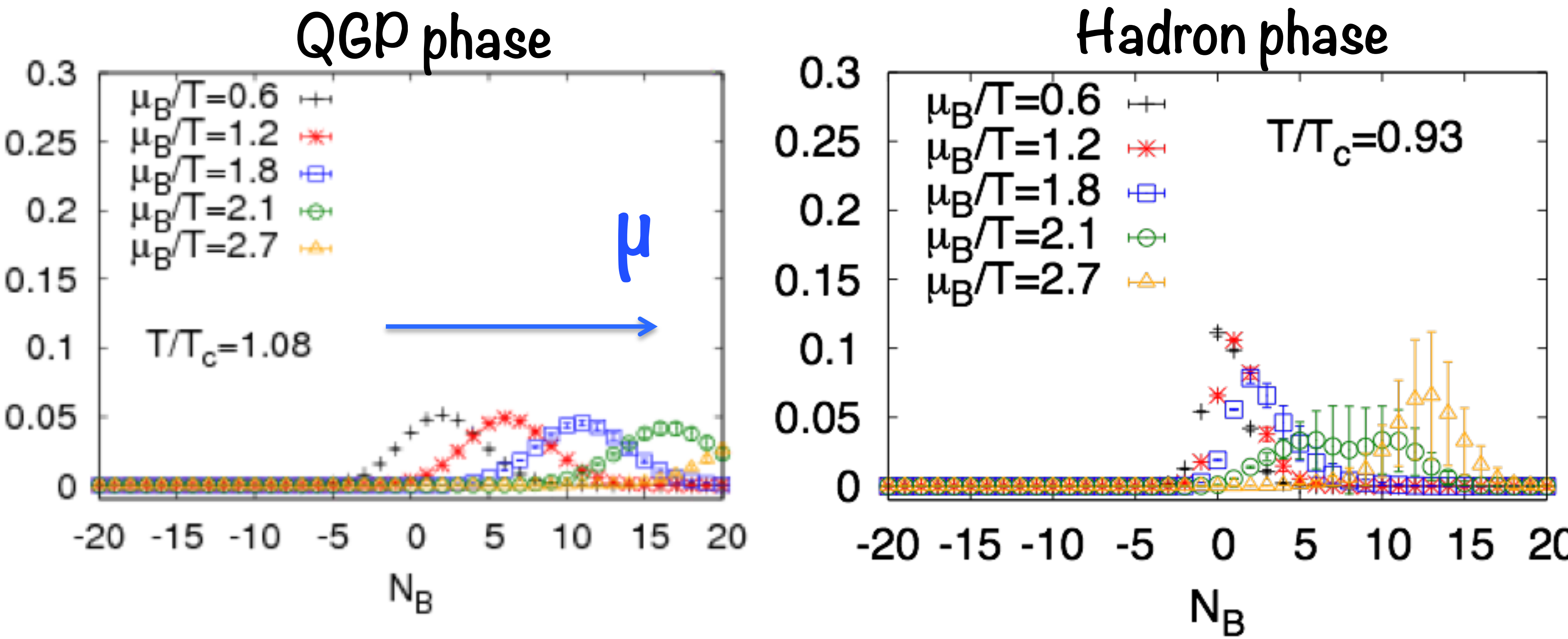
Zn can be obtained at μ=0

$$Z_n = \langle n | e^{-\beta \hat{H}} | n \rangle$$



KN, *et al* PTEP. 01A, 013 (2012);

$Z_{n_B} \exp(n_B \mu_B/T)$, (n_B: baryon number)



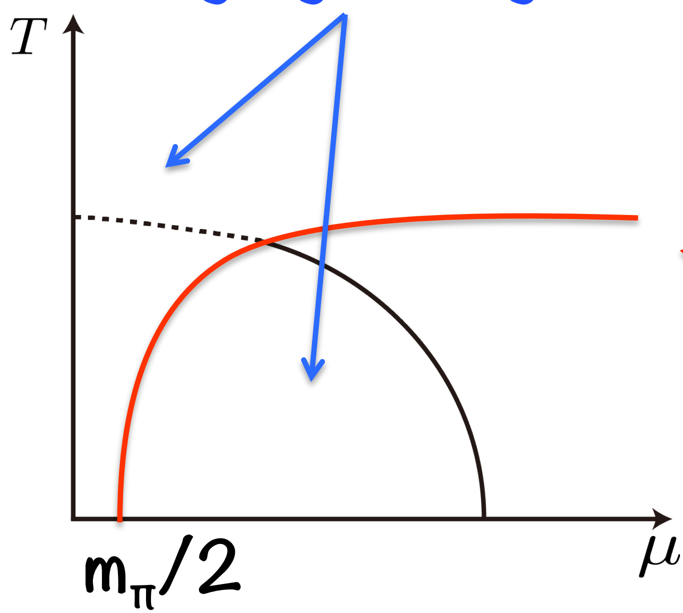
Non-monotonic behavior in hadron phase

=> more reliable estimation is needed for large n sector

Applicable limit

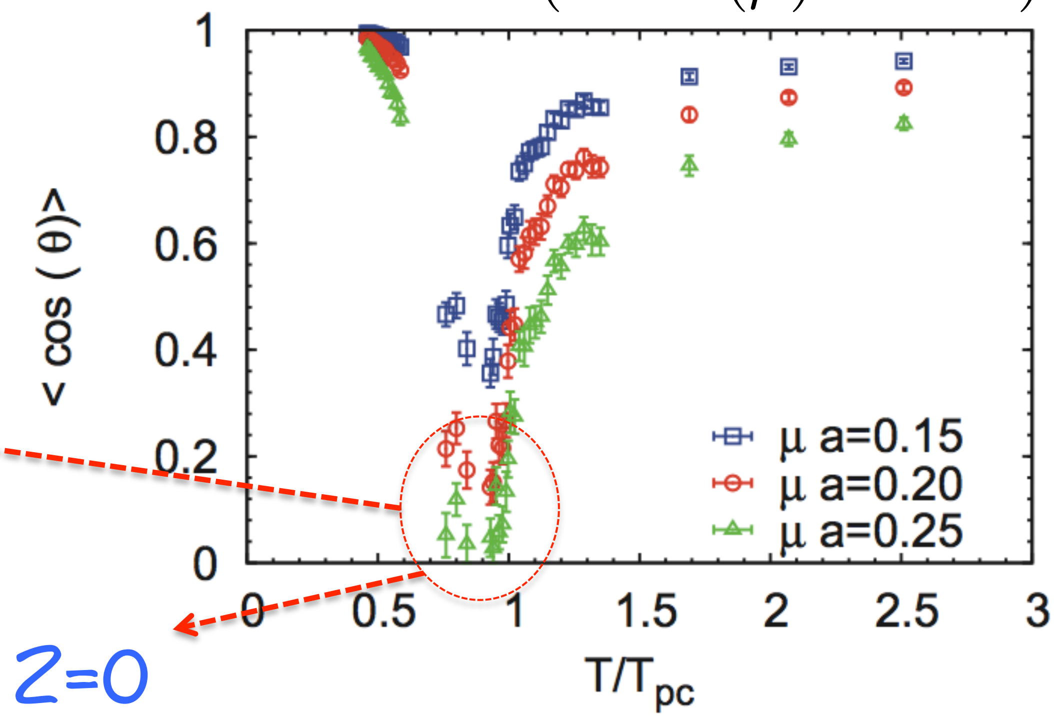
Overlap of gauge configs:
cf. microscopic states of ice and water

no-overlap of gauge configs.



Fluctuation of phase

$$(\det \Delta(\mu) \equiv re^{i\theta})$$



Z=0

KN et al PTEP01A013(2012).

Conventional approaches are limited to small μ.

- larger μ, larger systematice errors
  - e.g. slow convergence of Taylor expansion
  - other approaches also have similar problem
  - less overlap

MC-based finite density LQCD for nonzero μ
= real world + Lattice artifacts(finite size, etc) +
unknown systematic errors

CLM TO QGP PHASE
WHERE WE CAN STUDY

Developing new approaches has already begun

- e.g., complex Langevin method(CLM), Lefshetz thimble, Tensor network, etc
  - *based on ideas different from importance sampling*
  - some theories, which were out of scope of the conventional approaches, have already solved.
  - e.g chRMT at finite density

- *Among them, CLM has already applied to finite density QCD.*

(Real) Langevin method

- solve the path integral using the Langevin equation

[Parisi-Wu('81)]

$$Z = \int \prod dx_i \, e^{-S(x)}$$

$$\frac{\partial x_i}{\partial t} = -\frac{\partial S}{\partial x_i} + \eta_i(t)$$

t: Langevin time
η : Gaussian noise

- expectation value of O ~ Lagenvin time average

$$\langle O \rangle_{LM} = T^{-1} \int_0^T dt O(t)$$

- Validity of LM is proved using the Fokker-Plank eq.

$$\langle O \rangle_{LM} = \langle O \rangle_{phys}$$

Complex Langevin method(CLM)

- LM is available even for complex action [Parisi, Klauder('83)]
  - because it is free from the probability interpretation of e^{-S}

- Langevin eq. for S complex

$$\partial_t x_i = -\partial_x S + \eta \quad \in \text{complex}$$

  - Use of LM to complex S requires to extend real variables to complex [complexification]

$$x \in \mathbb{R} \to z = x + iy \in \mathbb{C}$$

  - extend action (also observables) analytically

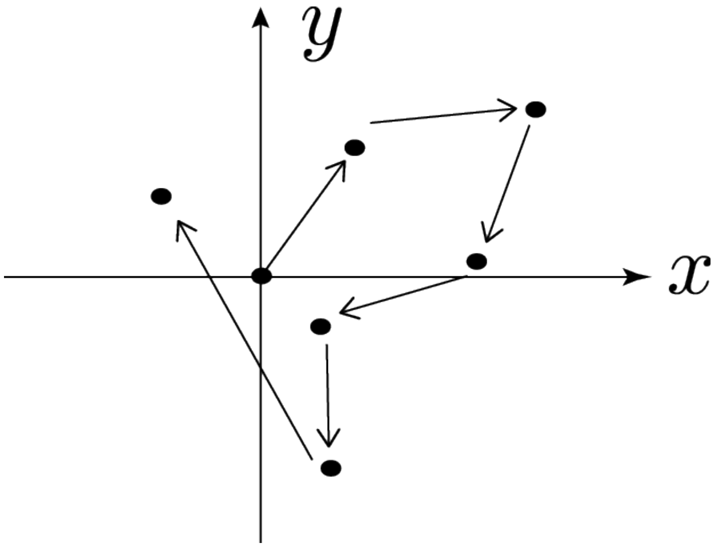
$$S(x) \to S(z) = S(x + iy)$$

CLM

- apply Langevin equation to z

$$\frac{\partial z}{\partial t} = -\frac{\partial S}{\partial z} + \eta(t)$$

  - configs. move on extended phase space



random walk
on complexified space

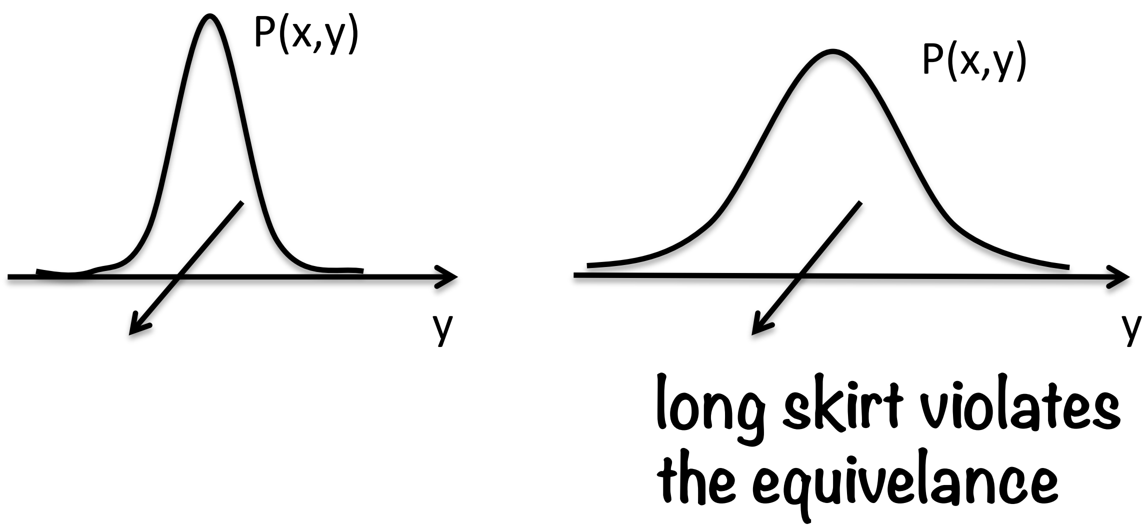
- expectation value = Langevin time average

$$\langle O \rangle_{CLM} = T^{-1} \int_0^T dt O(t)$$

Is CLM valid ?

$$\langle O \rangle_{CLM} \overset{?}{=} \langle O \rangle_{phys}$$

- equality holds if [Aarts, et. al. PRD81, 054508('10)].
  - *S* and *O* are independent of z*=x-iy [holomophy]
  - configs. do not extend to y-direction



Application to LQCD at nonzero μ

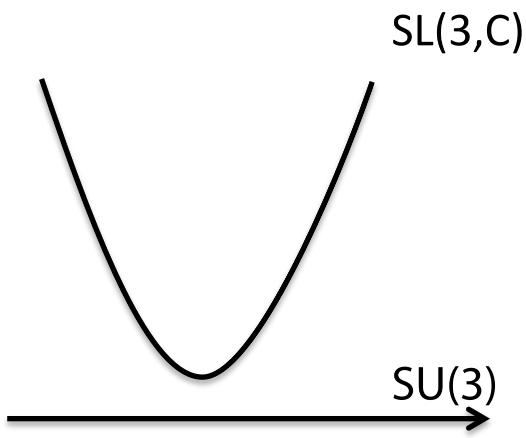
- complexification
  - link variables $U_{n,\mu} \in \mathrm{SU}(3) \to \mathcal{U}_{n,\mu} \in \mathrm{SL}(3,\mathbb{C})$
  - action and observables are analytically continued in a holomorphic manner
  - gauge invariance is also extended $\mathcal{U}_{n,\mu} \to g_n \mathcal{U}_{n,\mu} g_{x+\hat{\mu}}^{-1}$
- Langevin equation

$$\mathcal{U}_{n,\mu}(t+\epsilon) = e^{iX}\mathcal{U}_{n,\mu}(t),$$

$$X = \sum_a \lambda_a [(\mathcal{D}_{an\mu} S)\epsilon + \sqrt{2\epsilon}\eta_{an\mu}]$$

wrong convergence problem in QCD

- Excursion problem
  - Langevin dynamics is unstable in the complexified direction of link variables



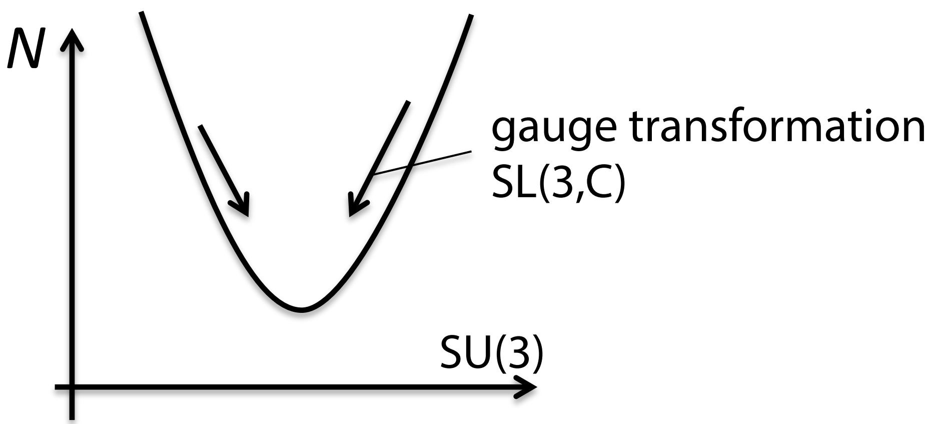
  - this spoils a validity condition (P(x,y)->0 for large y)

Gauge cooling [Seiler, et al.('12), Sexty ('14)]

- unitarity norm = "distance" from SU(3) matrices

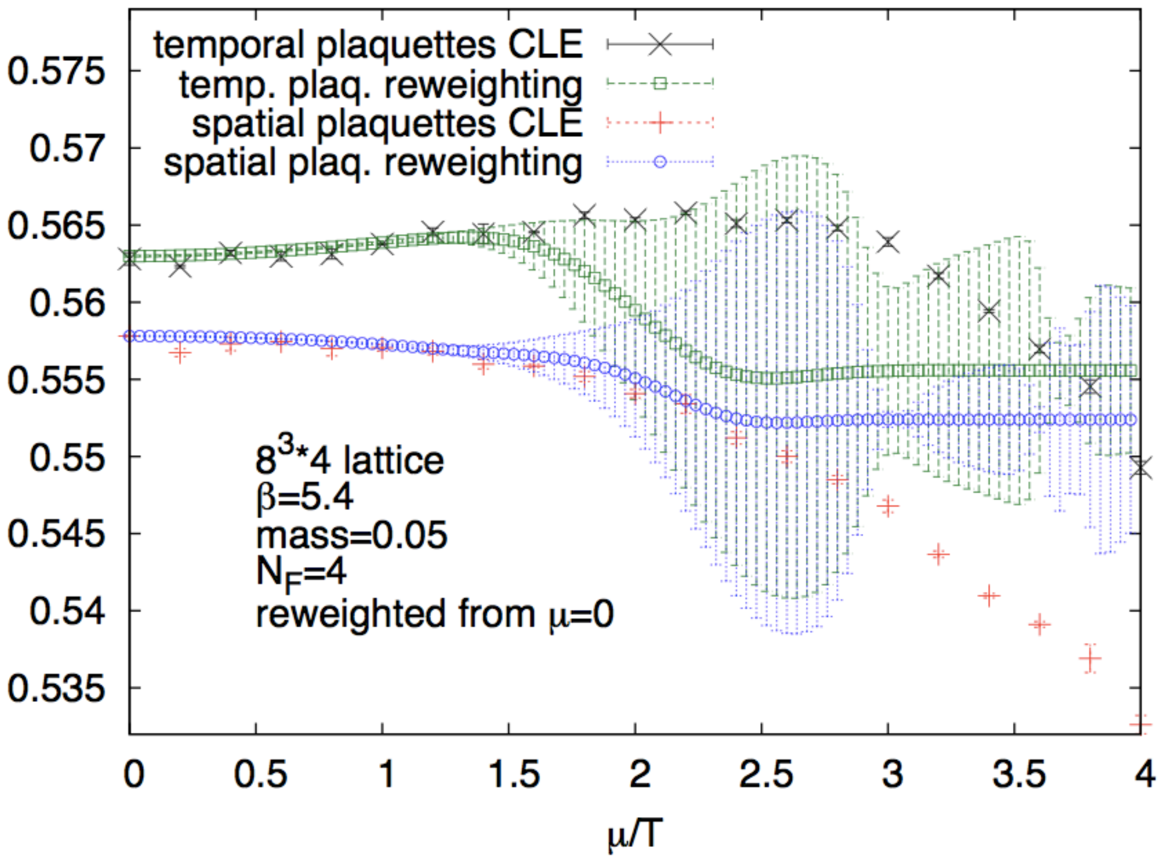
$$\mathcal{N} = \sum_{n,\mu} \mathrm{tr}[\mathcal{U}_{n,\mu}^{\dagger}\mathcal{U}_{n,\mu} + (\mathcal{U}_{n,\mu}^{\dagger})^{-1}\mathcal{U}_{n,\mu}^{-1} - 2]$$

- perform SL(3,C) trans. after every Langevin step



  – proof of validity of gauge cooling [KN, Shimasaki, Nishimura, 1508.02377]

CLM to QCD in QGP phase



Fodor, Katz,
Sexty, Toeroek('15)

- Simulation at μ/T ~ 4, far beyond μ/T = 1

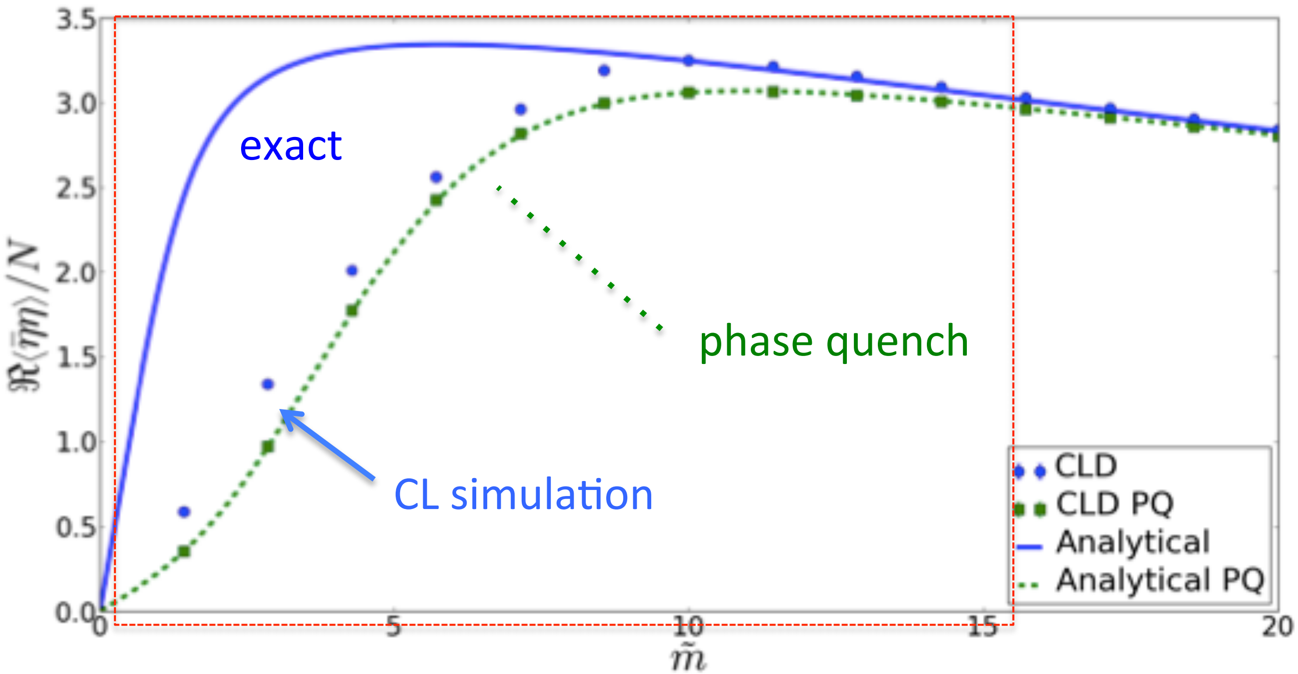
CLM TO HADRON PHASE
WHAT IS THE DIFFICULTY ?

Is CLM available to hadron phase ?

- another problem has been reported in some models

Chiral condensate vs quark mass in a ChRMT

[Mollgaard, Splittorff, PRD88 (2013), 11,116007]



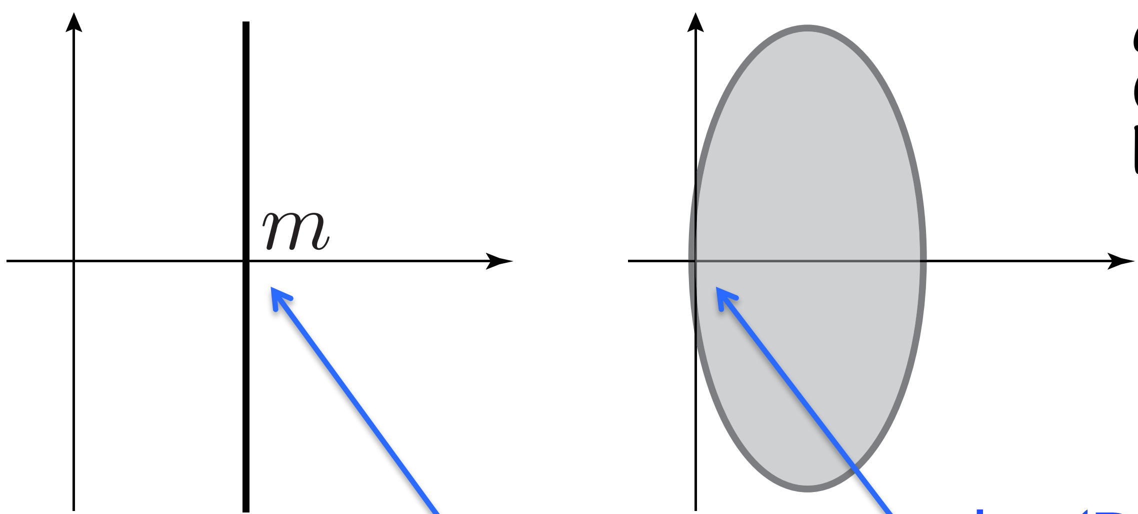
T=0
μ = fixed

What happens in hadron phase ?

- Fermion matrix, *D+m,* has zero eigenvalues at nonzero μ, which *violates holomorphy of S*

Distribution of eigenvalues of D+m at T=0
left: μ=0, right: nonzero μ

Related to
early onset problem in QCD
[KN et al. PTEP(2012)]



no gap in hadron phase
[Banks-Casher relation]

det (D+m) = 0
action is not holomorphic

Strategy

- Reducing lattice size [This talk]
  - zero-modes do not appear due to finite size effect

  - We introducenew criterion of correctness of CLM, and confirm the validity of the numerical results.

- on-going
  - generalization of gauge cooling(shown to work for RMT) [KN, Shimasaki, Nishimura(2016)]
  - deformation of action

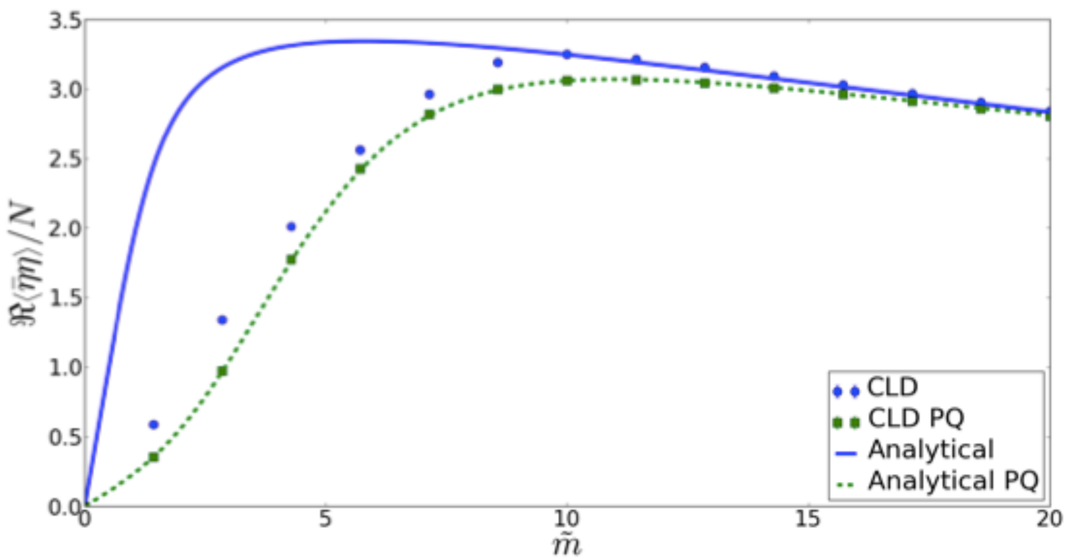
Simulation setup

- Lattice setup
  - size : Nx=Ny=Nz=4, Nt = 8, β= 5.7
  - quark mass: ma = 0.05, ($m_\pi a/2 \sim 0.27$ at mean field analysis)
  - μa = 0 ~ 2
  - Staggered fermion + plaquette gauge

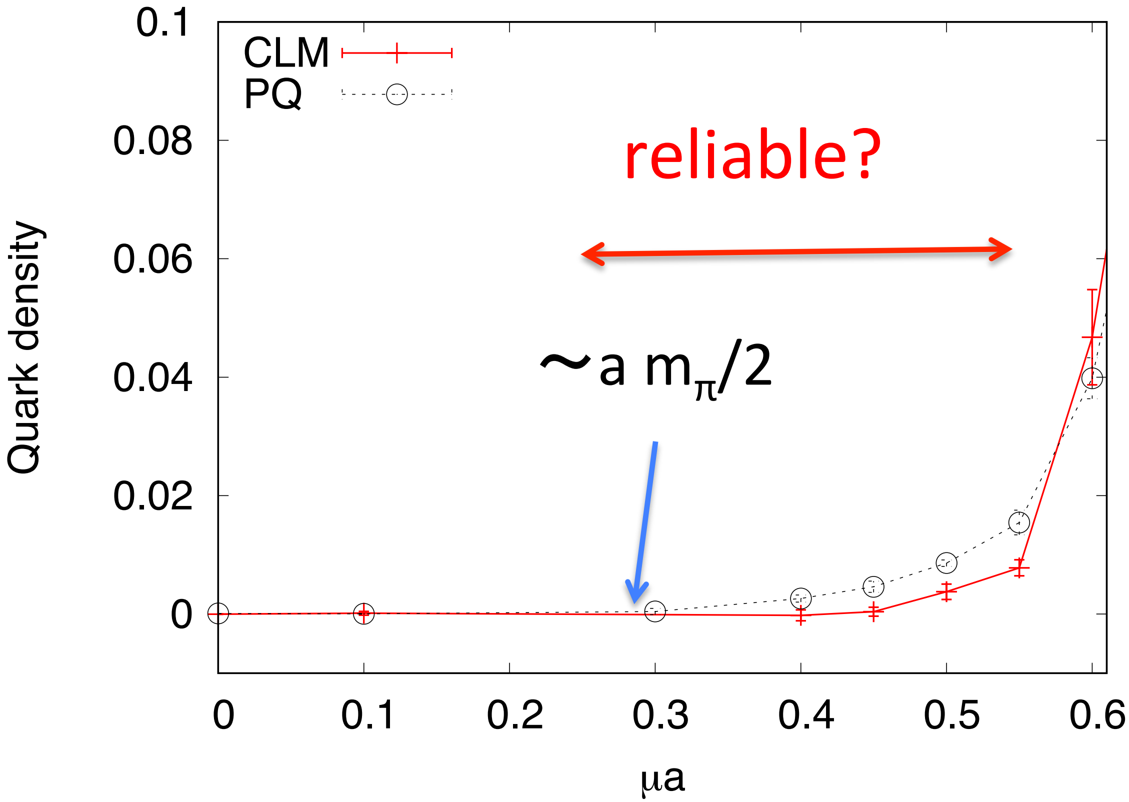
- Langevin setup
  - dtau = 10^{-4}, total Langevin time = 20~50 (preliminary)
  - gauge cooling max 50 steps after every Langevin step

Typical behavior when CLM fails
CLM = phase quanch

$$\det \Delta(\mu) \to |\det \Delta(\mu)|$$



Quark number density (CLM vs Phase Quench)



- onset of quark density
  - μ ~ 0.3 (PQ)
  - μ ~ 0.4 (CLM)

new criteria for correctness

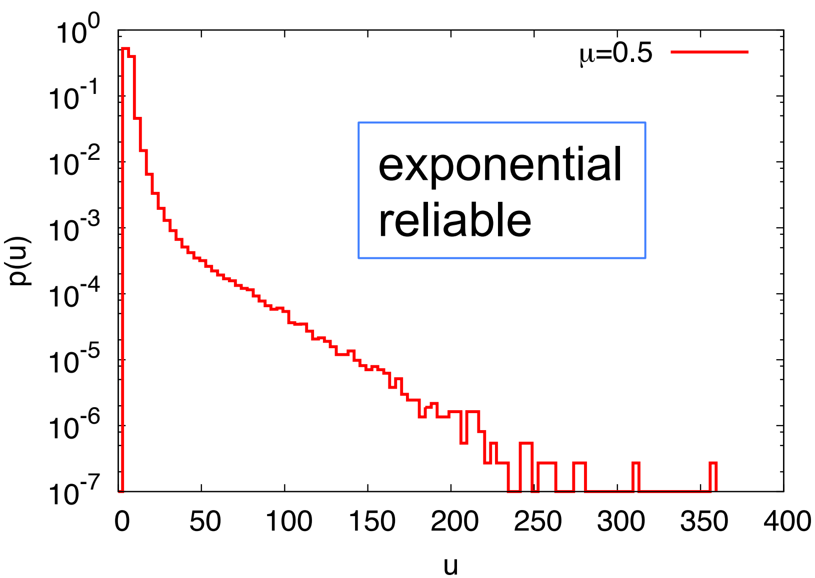
KN, Nishimura, Shimasaki, arXiv:1606.07627

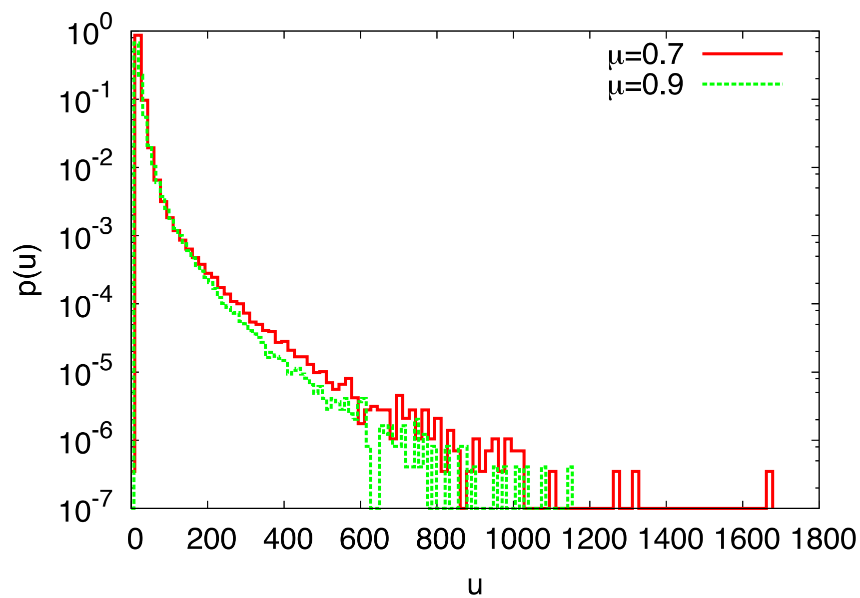
- probability distribution of drift

$$p(u;t) = \int \mathcal{D}\mathcal{U} \sum_{n,\mu} \delta\left(u - u_{n,\mu}(\mathcal{U})\right) P(\mathcal{U};t),$$

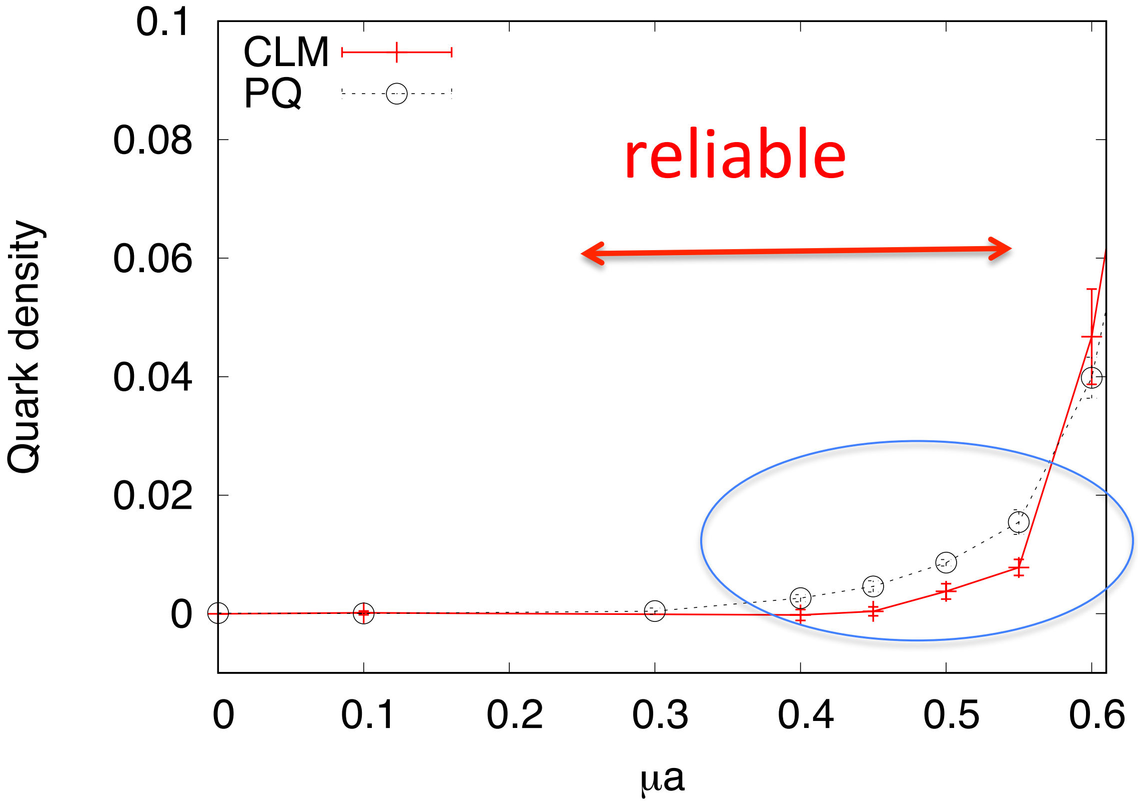
$$u_{n,\mu} = \sqrt{(N_c^2-1)^{-1} \sum_a v_{an\mu} v_{an\mu}^\dagger}$$

larger u:
near singularity
large excursion





Phenomenological understanding

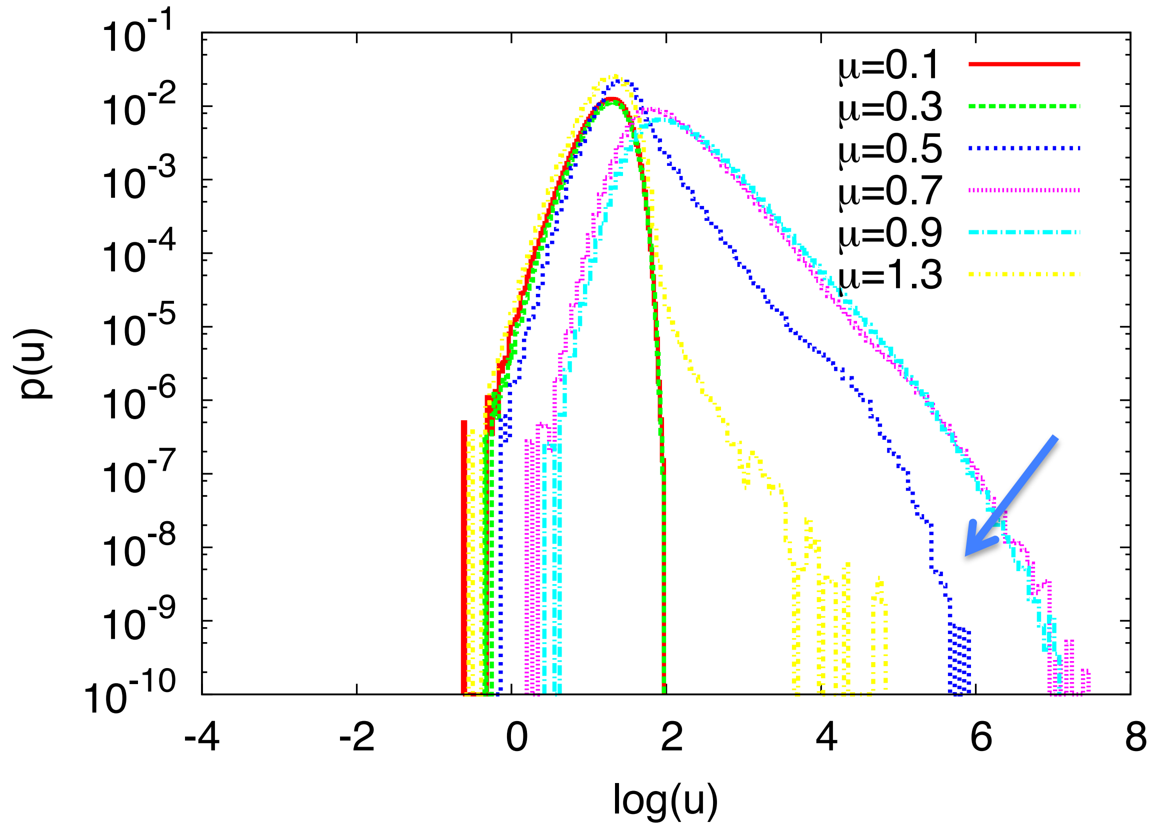


$$\rho = \rho_{\mathrm{NM}} \sim 10\rho_{\mathrm{NM}}$$

Summary: progress in fd LQCD

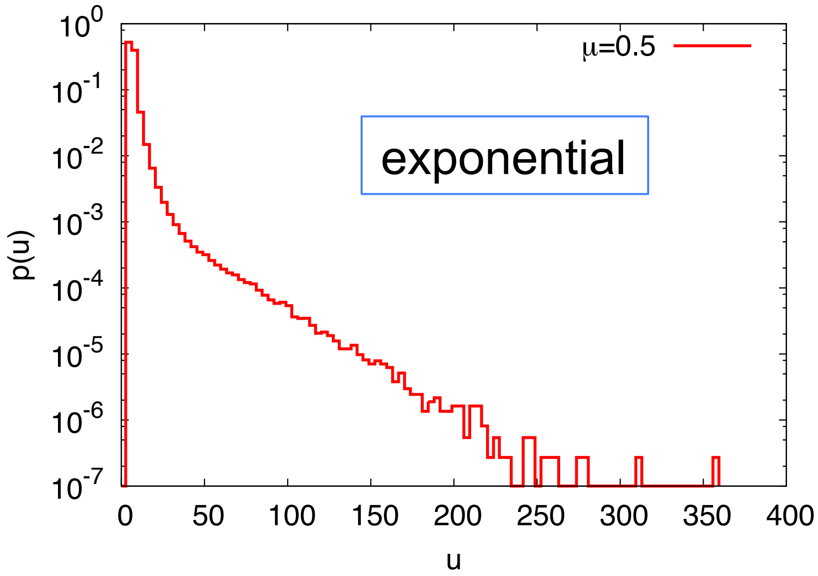
- MC-based approaches
  - pseudo critical line, EoS,cumulants
  - only for small density near Tc
- Development of new approaches has begun
  - CLM, Lefshetz thimble, tensor networks, etc
- CLM + gauge cooling
  - allows us to study *high density* in QGP phase
- Hadron phase is still challenging
  - singular drift problem

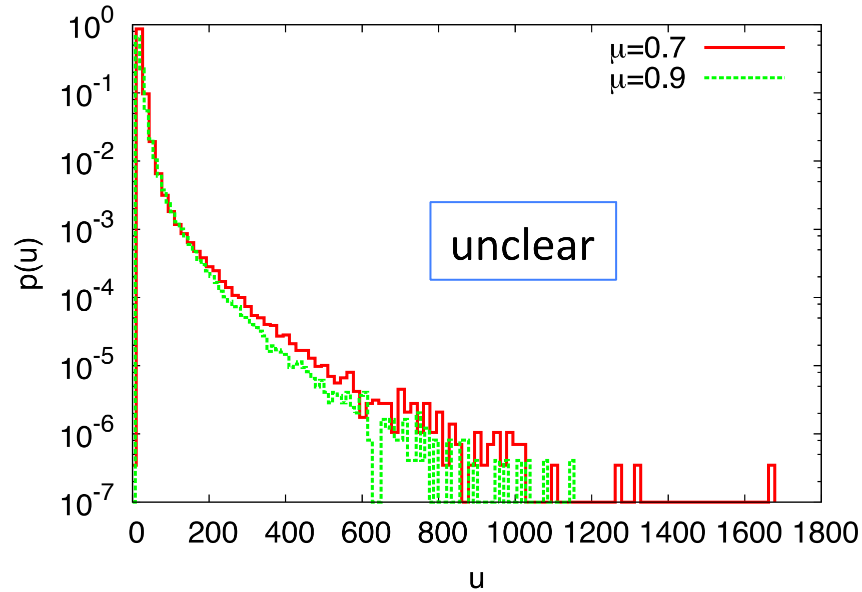
Probability of drift terms

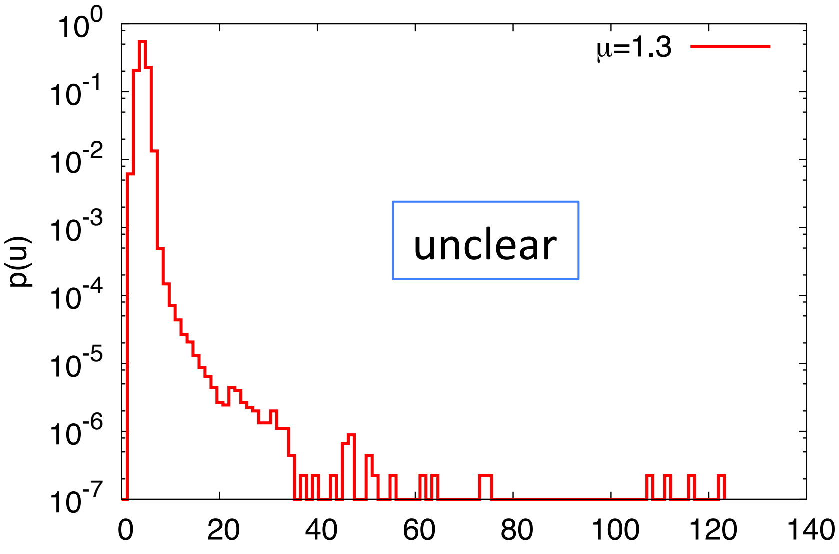


- μ≦0.3 : fall-off exponentially or faster => reliable
- how is for large mu ?

Data in semi-log plot







Proof of correctness of LM (BU)

- Average of an observable in LM is given by

$$\langle O(x^{(\eta)}(t))\rangle_\eta = \int dx\, O(x)P(x;t)$$

$$P(x;t) = \left\langle \prod_k \delta(x_k - x_k^{(\eta)}(t)) \right\rangle_\eta$$

$$\langle \cdots \rangle_\eta = \frac{\int \mathcal{D}\eta \cdots e^{-\frac{1}{4}\int d\tau \eta^2}}{\int \mathcal{D}\eta e^{-\frac{1}{4}\int d\tau \eta^2}}$$

- According to the Fokker-Planck equation, P converges to

$$\lim_{t\to\infty} P(x;t) \propto e^{-S(x)}$$

- Average of the observable converges to

$$\lim_{t\to\infty} \langle O(x(t))\rangle_\eta = \lim_{t\to\infty} \int \prod dx_k\, O(x)P(x;t),$$

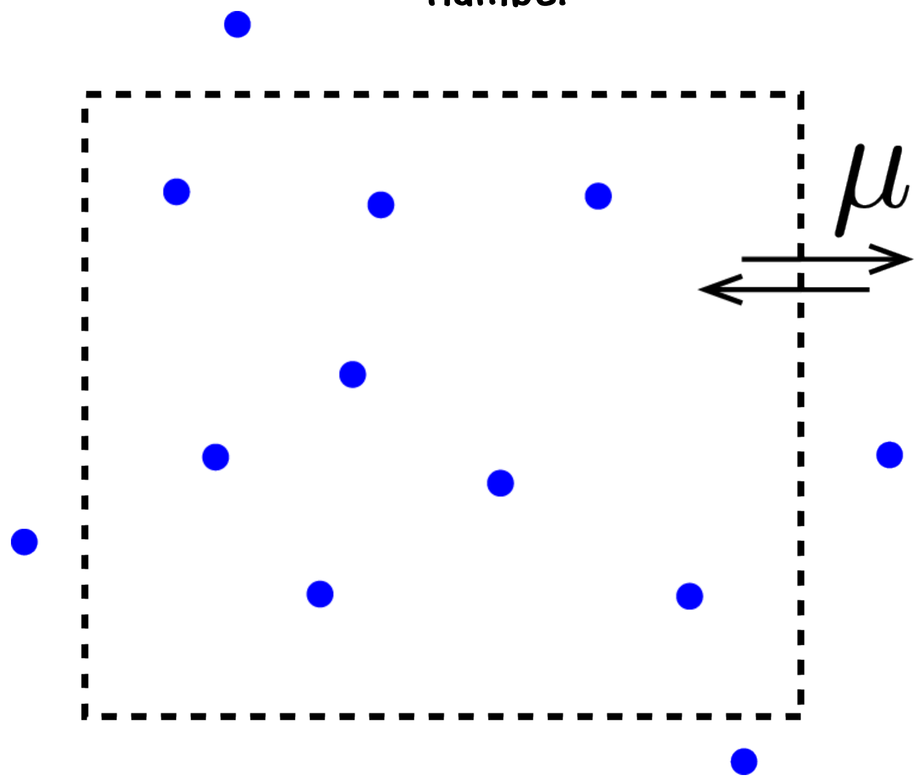
$$\propto \int \prod_k dx_k\, O(x)e^{-S(x)}$$

average in LM

physical expectation value

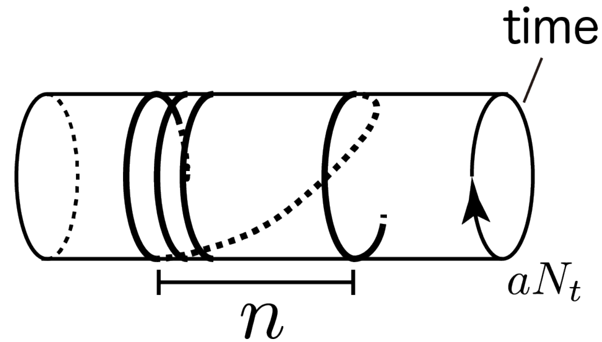
Canonical approach

$$\begin{pmatrix}\text{Grand canonical} \\ \text{ensemble}\end{pmatrix} = \sum_{\substack{\text{particle} \\ \text{number}}} \begin{pmatrix}\text{Canonical} \\ \text{ensembles}\end{pmatrix}$$

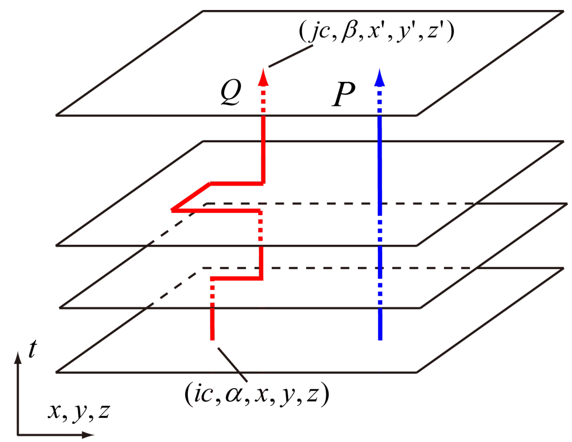


Canonical approach: observable on lattice

$Z_n = \sum_{\text{loops \& ensembles}}$ closed loops with temporal winding number n

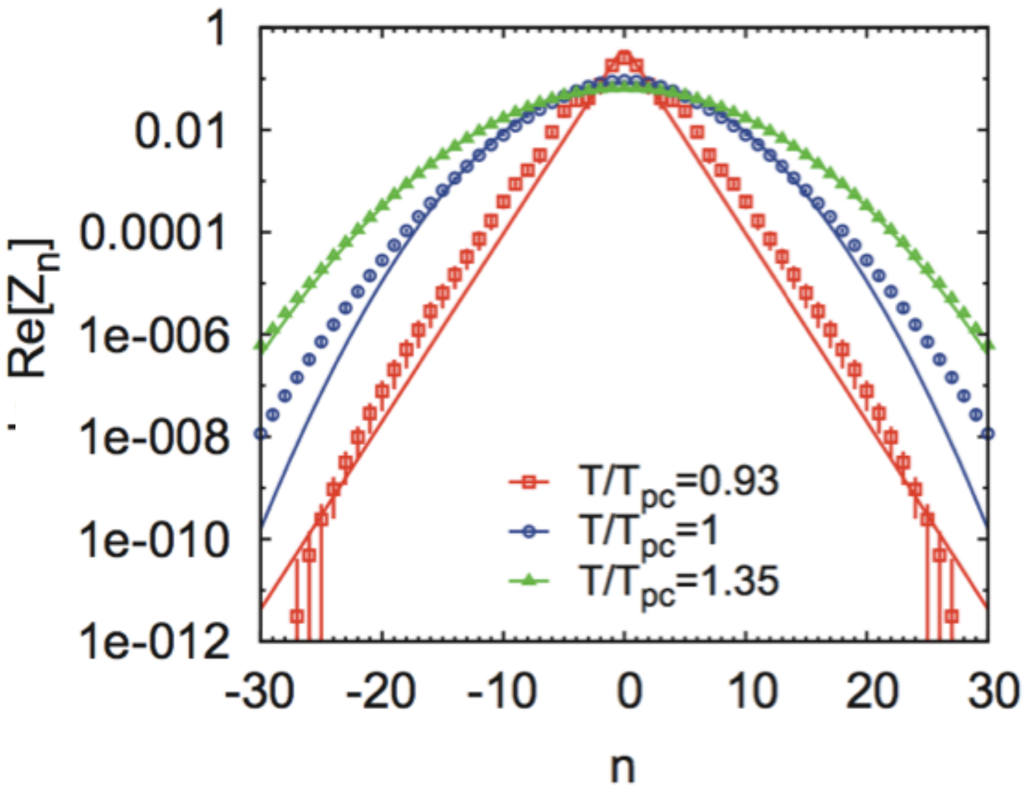


- Reduction of fermion determinant
  - Q : transfer matrix
  - Q^n : winding number n



Reduction formula [Gibbs, PLB 172, 53 ('86). Hasenfratz &Toussaint, NPB371, 539('92), Borici, PTP. Suppl. 153, 335 ('04). Alexandru &Wenger, PRD83, 034502 ('11). KN&AN, PRD82,094027 ('10). Adams, PRL92, 162002 ('04), PRD70, 045002 ('04).]

Numerical result



Setup: clover-improved Wilson fermion,
RG-improved gauge action
Nf = 2
mpi ~ 800 MeV
Nx=Ny=Nz= 8, 10, Nt = 4

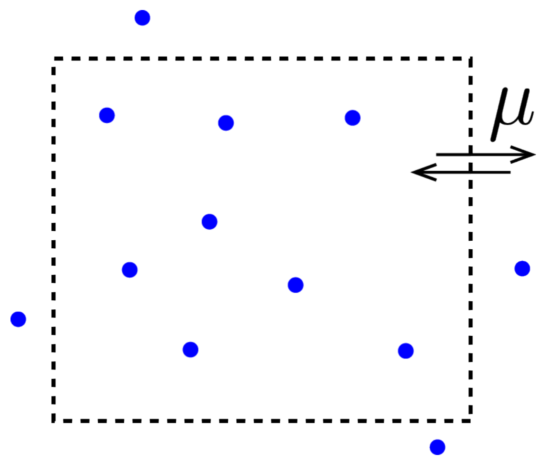
gauge configurations at μ=0 are used (reweighting)

KN, S. Motoki, Y. Nakagawa, A. Nakamura, and T. Saito, PTEP. 01A, 013 (2012); arXiv:1204.6480.

From Gaussian to non-Gaussian as T decreases

possible criticisms

Sampling at μ = 0:
large n component is suppressed exponentially



Systematic errors in tail of the distribution (due to finite # of statistics)

Careful analysis is necessary to conclude the finding the phase transitions.